Outer space and Automorphisms of free groups

LECTURE 2

We were considering the question:

$F_n = F\langle a_1, \ldots a_n\rangle$, $\{w_1, \ldots, w_n\}$ distinct elements of F_n

when is $a_i \longmapsto w_i$ an automorphism?

eg
$$\begin{cases} a \longmapsto ab \\ b \longmapsto ba^{-1}bc^{-1} \\ c \longmapsto cab \end{cases}$$

Is this an automorphism?

Whitehead gave an algorithm for answering the question, using $M_n = \# S^1 \times S^2$ as a model for F_n, homeomorphisms as a model for automorphisms

Nielsen had shown that the following automorphisms of $F\langle a_1, \ldots, a_n\rangle$ generate $\mathrm{Aut}\, F_n$ (and hence $\mathrm{Out}(F_n)$):

$$\rho_{ij} : \begin{cases} a_i \longmapsto a_i a_j \\ a_k \longmapsto a_k & k \neq i \end{cases}$$

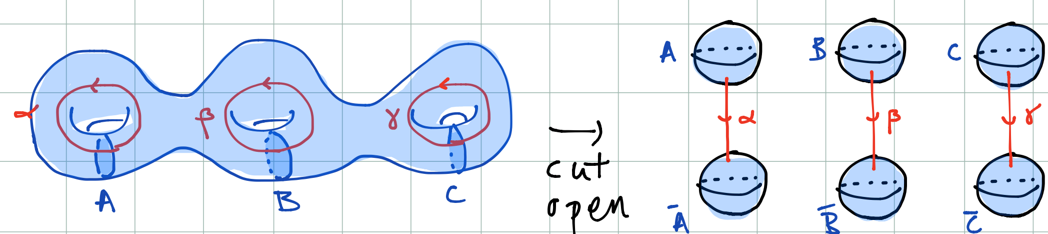
$$\lambda_{ij} : \begin{cases} a_i \longmapsto a_j a_i \\ a_k \longmapsto a_k & k \neq i \end{cases}$$

$\varepsilon_1 : a_1 \longmapsto a_1^{-1}$ generate $\mathrm{Aut}(F_n)$ ($\therefore$ hence $\mathrm{Out}(F_n)$)

Assume this for now

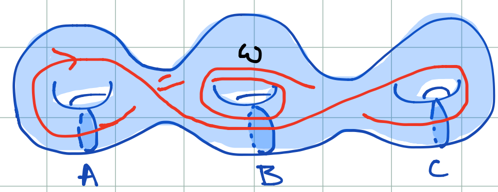
<u>Claim</u> $\pi_0 \, \mathrm{Homeo} \, M_n \longrightarrow \mathrm{Out}(F_n)$ is surjective

By Nielsen, it
Suffices to realize $\rho_{ij}, \lambda_{ij}, \varepsilon_1$ on M_n.



$\times 2$

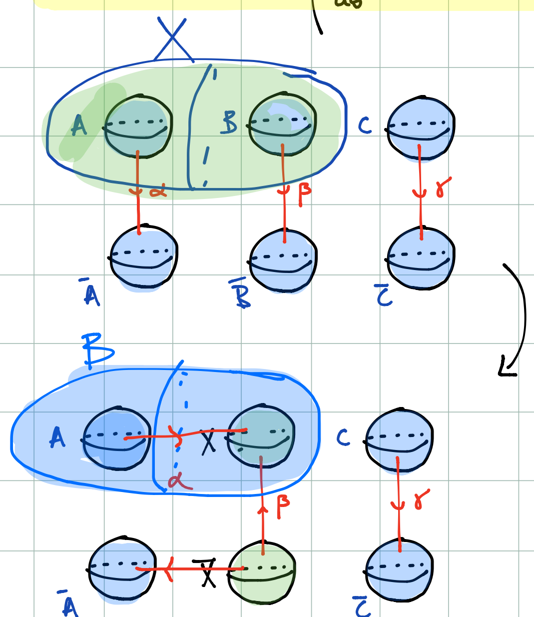
α, β and γ generate $\pi_1(M_n)$ (we're ignoring basepts)
w another loop ($=$ elt of $\pi_1 M_n$)



$w = b c^{-1} b a^{-1}$

you can read off w as a cyclic word in F_3 by looking at which spheres it punctures (in which direction

To realize $\rho_{ab}: a \mapsto ab$



Choose a diffeomorphism sending $A \to A$, $B \leftrightarrow X$, $C \to C$

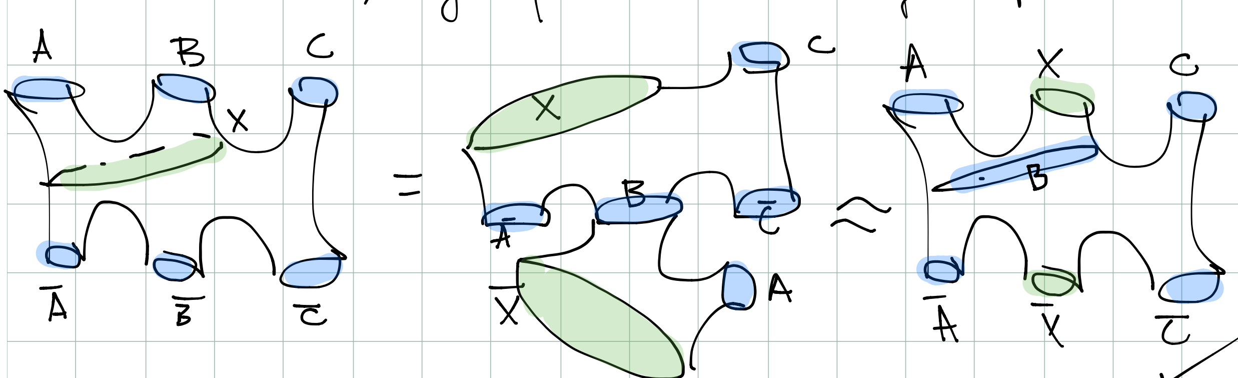
"X is the new B"

This sends $a \mapsto ab^{-1}$

$b \mapsto b^{-1}$

$c \mapsto c$

To see the diffeo, may help to look at half the picture:

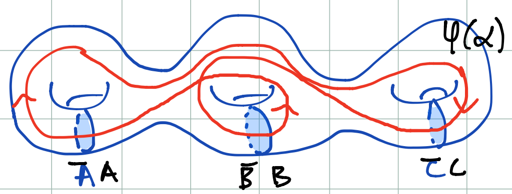


Now look at a general automorphism:

Let $\varphi \in \mathrm{Out}\, F_n$, represent φ by a diffeomorphism of M_n

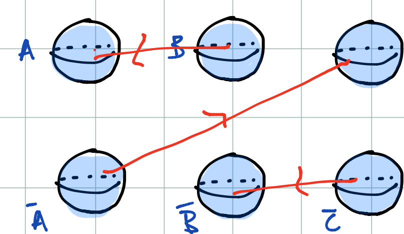
Image of α is a loop

one view:



$\varphi(a) = c^{-1} b a^{-1}$

another view:



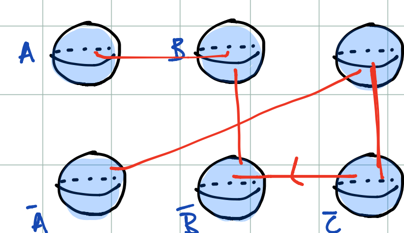
Looks like a graph if you squint

Called the <u>star graph</u> of $\varphi(a)$.

Put $\varphi(a), \varphi(b), \varphi(c)$ in the same graph

= star graph $St(\varphi)$

<u>eg</u> $\varphi(b) = b$
$\varphi(c) = c$



① <u>Notes</u>: Valence (X) = valence $(\bar{X})$ = # of occurrences of the letter X in the cyclically reduced w_i.

② $St(\varphi)$ uses all vertices:

If it misses A (and $\bar{A}$) no w_i contains a or a^{-1} so $\{w_i\}$ isn't a basis

Lemma φ an automorphism $\Rightarrow$ $St(\varphi)$ has a cut vertex or is disconnected

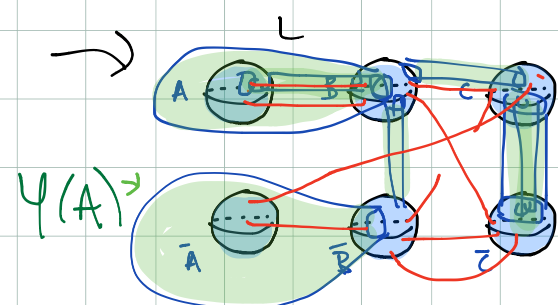
Pf: Put the image $\varphi(A)$ in the picture as well:

First Suppose $\varphi(A)$ intersects some $X = A, B$ or C

Make the $\cap$'s with A B & C transverse, so $\varphi(A) \cap (A \cup B \cup C)$ is a union of circles. These circles cut $\varphi(A)$ into planar surfaces, including ≥ 2 disks.

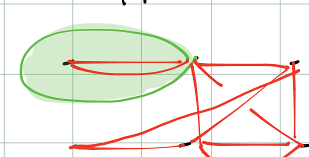
$St(\varphi) = \varphi(\alpha) \cup \varphi(\beta) \cup \varphi(\gamma)$ intersects $\varphi(A)$ in only one point, so <u>one of the disks is missed</u>!

This disk misses $St(\varphi)$ $\rightarrow$



The disk intersects exactly one X or $\bar{X}$, corresp to a a vertex of $St(\varphi)$. Other vertices are on one side or the other of the disk, and the star graph doesn't connect any vertices on opposite sides

ie X is a cut vertex for $St(\varphi)$:

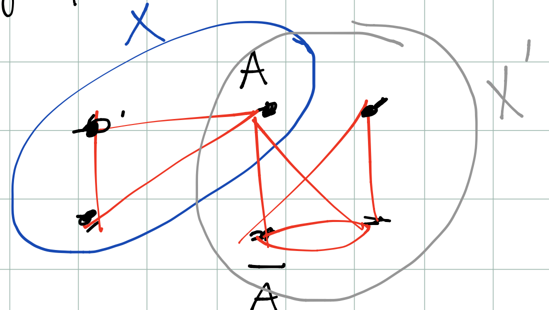


Exercise: What happens if $\varphi(A)$ doesn't intersect any X?

Now for the algorithm:
Define the complexity of $\varphi = \sum_{v \in St(\varphi)} \text{valence}(v)$

Claim I can find a diffeomorphism of M_n which decreases the complexity of the star graph.

pf: Let X be a sphere separating the star graph at the cut vertex:



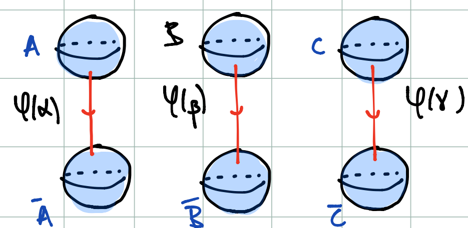
We can assume X separates A from $\bar{A}$: (if not, use the other sphere)
Choose a diffeomorphism sending $A \leftrightarrow X$, fixing all other spheres

The old star graph intersects X less than it intersects A. So the new star graph has lower complexity. ✓

The algorithm: Keep reducing complexity

Minimal complexity $= 3$: $\varphi(\alpha)$ has to intersect some A, B, or C since it is a non-trivial loop in $\pi_1 M_n$

Same for $\varphi(b)$ and $\varphi(c)$



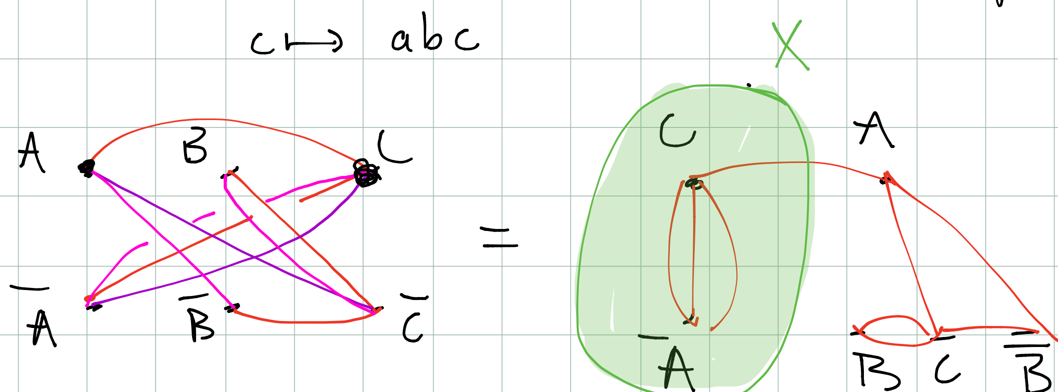
If it is exactly one in each case, then $\{\varphi(\alpha), \varphi(\beta), \varphi(\gamma)\} = \{\alpha^{\pm}, \beta^{\pm}, \gamma^{\pm}\}$ and φ could be an automorphism

<u>Example</u>: $a \mapsto a\bar{c}bc$

$b \mapsto ca$

$c \mapsto abc$

Is this an automorphism?



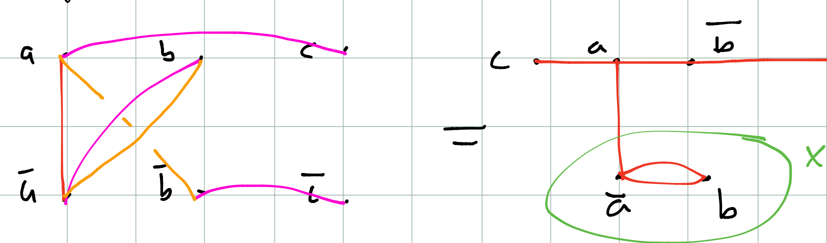
$(\bar{a}^1 \longmapsto \bar{a}^1 c)$ (replace C by X)

So $a \longmapsto c^{-1} a$ should shorten it.

$$\begin{aligned}
a &\longmapsto a\bar{c}bc &&\longmapsto c^{-1}ac^{-1}bc &&\sim ac^{-1}b \\
b &\longmapsto ca &&\longmapsto cc^{-1}a &&= a \\
c &\longmapsto abc &&\longmapsto c^{-1}abc &&\sim bc
\end{aligned}$$

complexity $\underline{\underline{6}}$

New star graph is



replace $a \leftrightarrow x$

corresps to $c \mapsto c$, $a \mapsto a$ $b \mapsto ba^{-1}$

$x \mapsto cxc^{-1}$ $b \mapsto ba^{-1}$ $x \mapsto a^{-1}xa$

$$\begin{aligned}
c^{-1}ac^{-1}bc &\longmapsto ac^{-1}b &&\longmapsto ac^{-1}ba^{-1} &&\to c^{-1}b &&= c^{-1}b \\
a &\longmapsto cac^{-1} &&\longmapsto cac^{-1} &&\to a^{-1}cac^{-1}a &&\sim a \\
c^{-1}abc &\longmapsto ab &&\longmapsto aba^{-1} &&\to b &&\sim b
\end{aligned}$$

$c \mapsto bc^{-1}$

$$\begin{aligned}
c^{-1}b &\longmapsto cb^{-1}b &&= c \\
a^{-1}cac^{-1}a &\mapsto a^{-1}bc^{-1}acb^{-1}a &&= a^{-1}bc^{-1}\,a\,cb^{-1}a \\
b &\longmapsto b &&= b
\end{aligned}$$

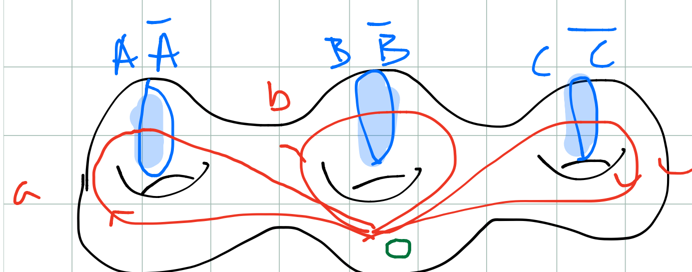
complexity $= \underline{\underline{3}}$

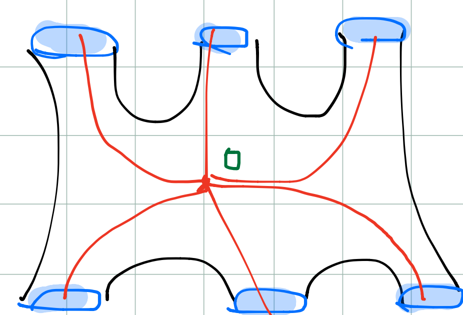
(minimal!

In this way you reduce to a map sending each generator to a conjugate of another generator. You then need to decide whether that is an automorphism.

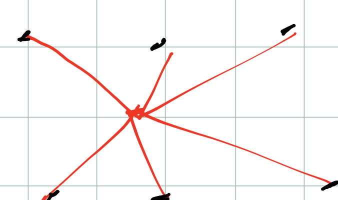
In our case this is not an automorphism!
Can't make "a" with this basis, since any word with 1 a in it has at least 3 a's in it.

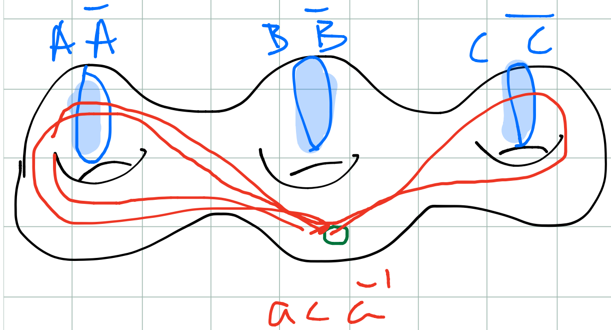
If you want actual automorphisms, need to put a basept in the star graph — slightly more complicated, but the same idea

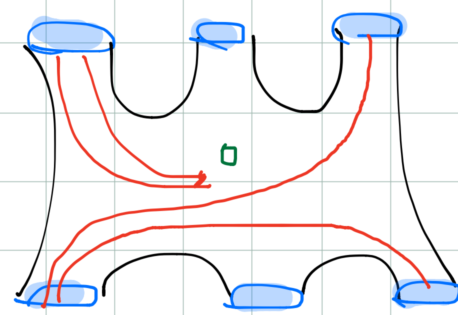




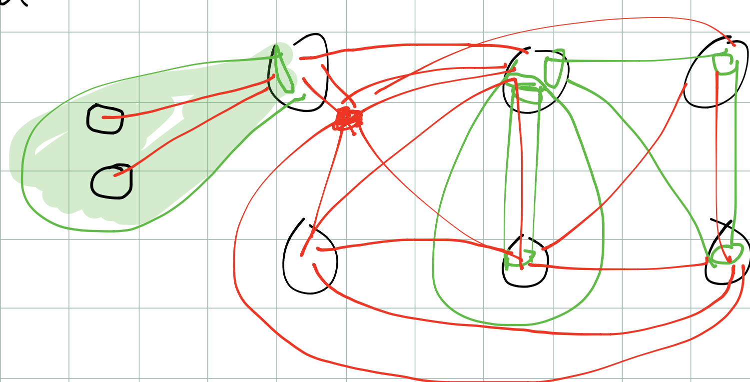
(Now it really does look like a star graph!)







Still true that $St^+(\varphi)$ intersects $\varphi(A)$ in one pt., and $\varphi(A)$ has at least two disk components, so one disk is missed



So graph $St^+\varphi$ has a cut vertex other than 0.

Whitehead refined this to an algorithm:
Given $u_1, \ldots, u_k$, $w_1, \ldots, w_k$ words in the a_i
Is there an automorphism φ with $\varphi(u_i) = w_i$?

<u>Exercise</u>: Use the Whitehead algorithm to decide whether these are automorphisms:

$$\begin{cases} a \mapsto ab \\ b \mapsto ba^{-1}bc^{-1} \\ c \mapsto cab \end{cases}$$

$$\begin{cases} a \mapsto ab \\ b \mapsto bcab \\ c \mapsto b^{-1}a^{-1}c \end{cases}$$

Now write down three random words w_1, w_2, w_3 and use Whitehead's algorithm.

Fast forward 40 years

What about Nielsen's theorem that the ρ_{ij}, λ_{ij} and ε_1 generate $\mathrm{Aut}(F_n)$?

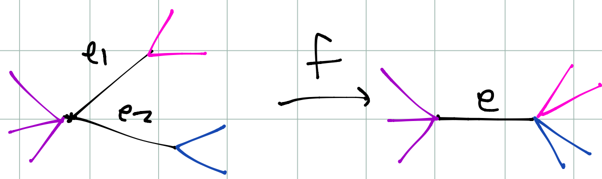
Stallings gave an elegant proof using the graph model for F_n. It also gives an alternate to Whitehead's algorithm.

Def X, Y graphs. A map $f: X \to Y$ is a graph morphism if

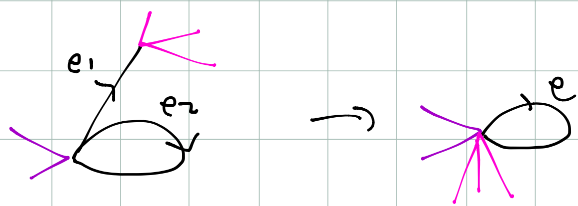
1. vertices $\mapsto$ vertices
2. Can subdivide edges of X so that each edge is either collapsed to a vertex or sent linearly to an edge of Y.

Def A Stallings fold is a graph morphism which identifies two edges emanating from the same vertex of X but makes no other identifications

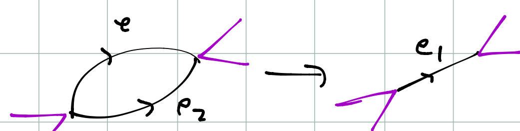
Examples:



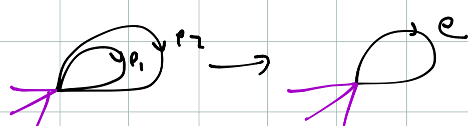
or



or:



or



singular folds

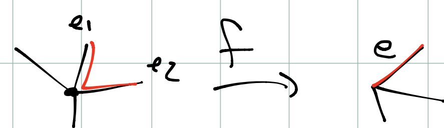
– reduce rk π_1

Non-singular folds are homotopy equivalences.

<u>Lemma</u>: If $f: X \to Y$ a graph morphism is not locally injective, then either

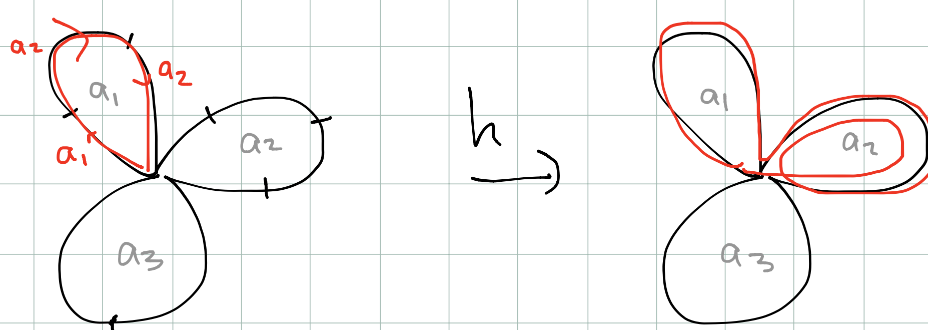
1. some edge collapses or
2. Two edges coming out of the same vertex have the same image

Pf Suppose X is subdivided, and no edge collapses
Linear on edges $\Rightarrow$ loc. injective on edges
Suppose not loc. injective at some vertex:



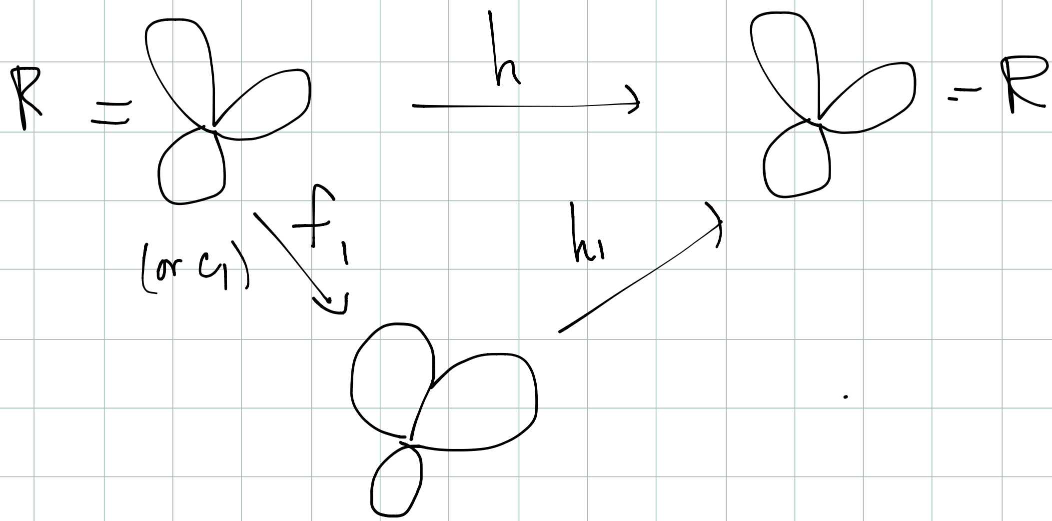
Now let $\varphi \in Out(F_n)$ be an (outer) automorphism
Represent it by a graph morphism which is a homotopy equivalence. h

eg $a_i \mapsto w_i$



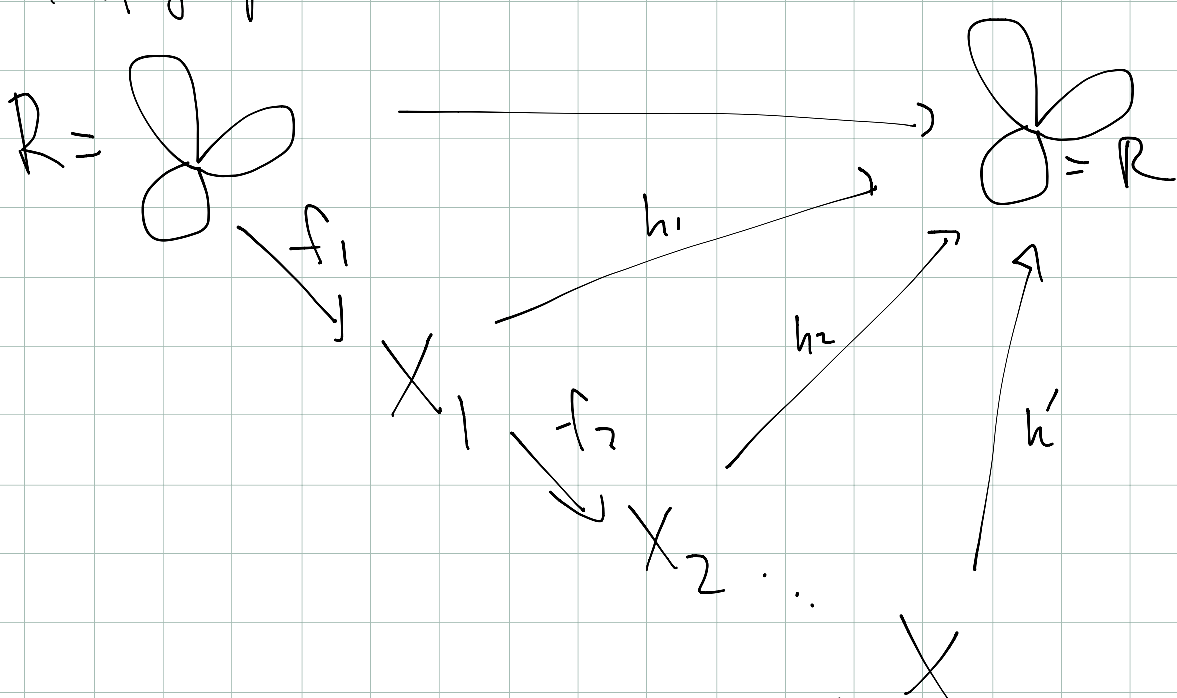
$a_1 \mapsto a_1 a_2^2$
$a_2 \mapsto w_2$
$a_3 \mapsto w_3$

If h is not locally injective, fold or collapse an edge.



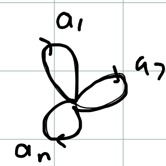
Get a new graph morphism h_1, also a homotopy equivalence. (f_* injective $\Rightarrow$ f is a h.e $\Rightarrow$ h_1 is a h.e)

Keep going:



until you get to a locally injective $h' : X \longrightarrow R$

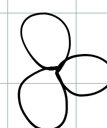
Claim h' is a homeomorphism

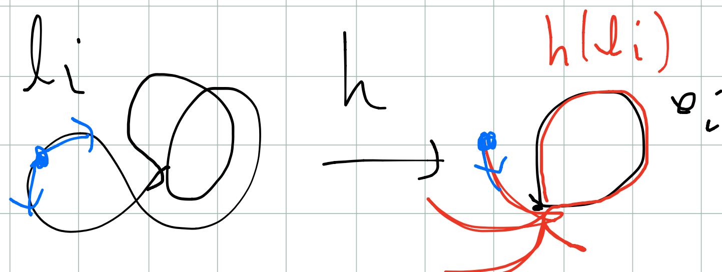
Pf: $X \xrightarrow{h'}$ 

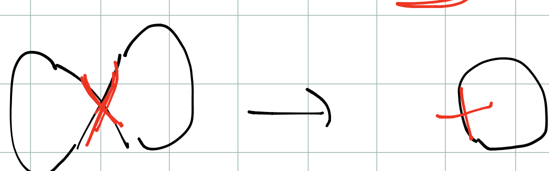
h' a homotopy equivalence $\Rightarrow$ some loop in X is sent to e_i : $h'(l_i) \simeq e_i$

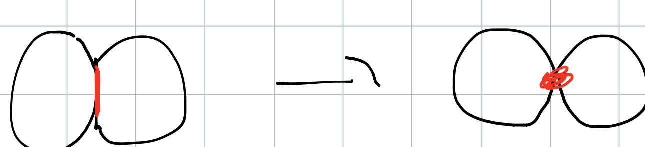
h locally injective $\Rightarrow$

- $h'(l_i) \subseteq e_i$
- l_i is simple
- $l_i \cap l_j$ contains no edges
- $\bigcup l_i = X$
- $l_i \cap l_j = pt$

$\Rightarrow X =$ 







etc.

Claim: This procedure gives a way of factoring φ as a product of $\rho_{ij}, \lambda_{ij}, \sigma, \varepsilon_i$

$$X_0 = R \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2 \longrightarrow \cdots \xrightarrow{f_r} X_k \xrightarrow[\approx]{\text{signed permutation}} R$$

Want to identify F_n with $\pi_1 X_i$ for all i

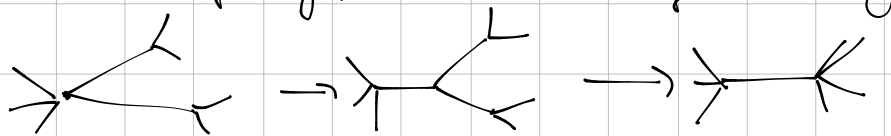
Usual way to do this: choose a maximal tree $T_i \subset X_i$, orient and label the rest of the edges with generators a_i of F_n

$$\begin{tikzcd}
X_0 = R \arrow[r, "f_1"] \arrow[d] & X_1 \arrow[r, "f_2"] \arrow[d] & X_2 \arrow[r] \arrow[d] & \cdots \arrow[r, "f_r"] & X_k \arrow[r, "\approx"] \arrow[d, "\approx"] & R \arrow[d, "\approx"] \\
R_n \arrow[r, bend left, "\varphi_1"] \arrow[rrrrr, bend right, "\varphi"'] & R_n \arrow[r, bend left, "\varphi_2"] & R_n \arrow[r, bend left, "\varphi_3"] & \cdots \arrow[r, bend left, "\varphi_r"] & R_n \arrow[r, "\approx"] \arrow[r, bend left, "h"] & R_n
\end{tikzcd}$$

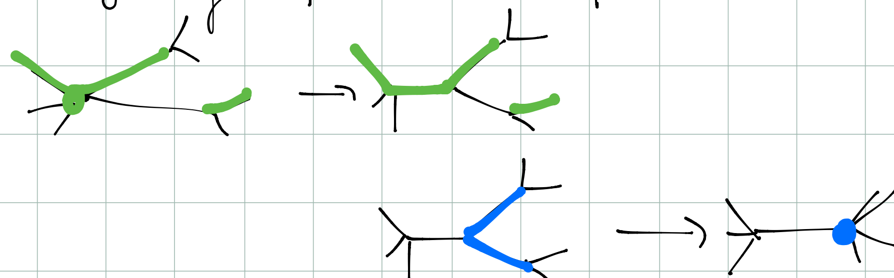
$\varphi = h \circ \varphi_r \circ \cdots \circ \varphi_1$

Each vertical arrow is "collapse T_i"

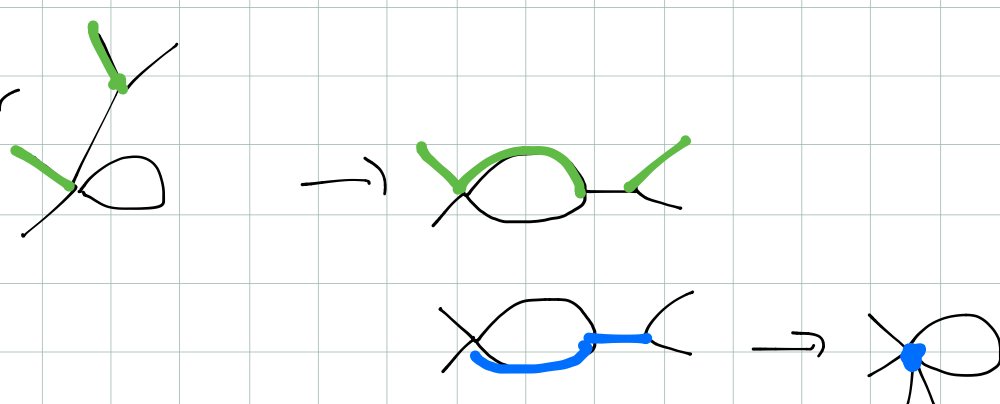
To keep track of what's happening, convenient to think of a fold as occuring in 2 stages: fold halfway, then the rest of the way



If you have a maximal tree in X, first half grows the tree, second half collapses 2 edges of X, neither a loop:



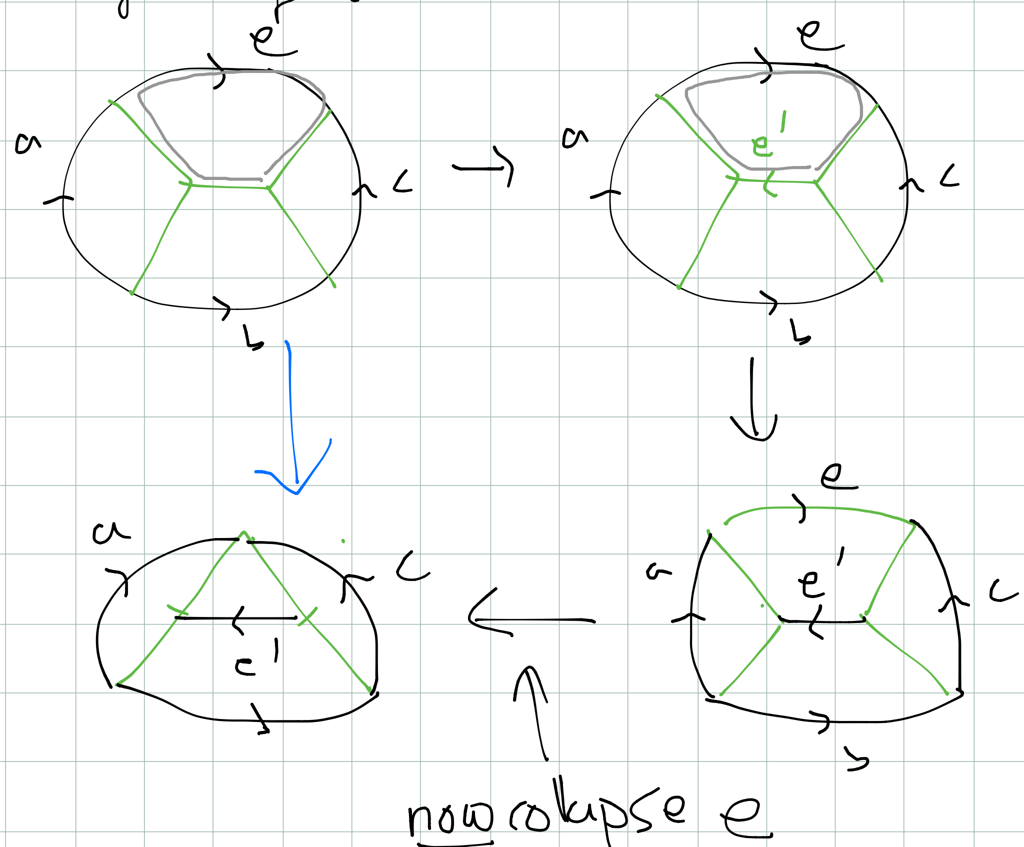
or



So, given a tree $T_i \subset X_i$, first grow it.
Then we want to collapse a couple of edges

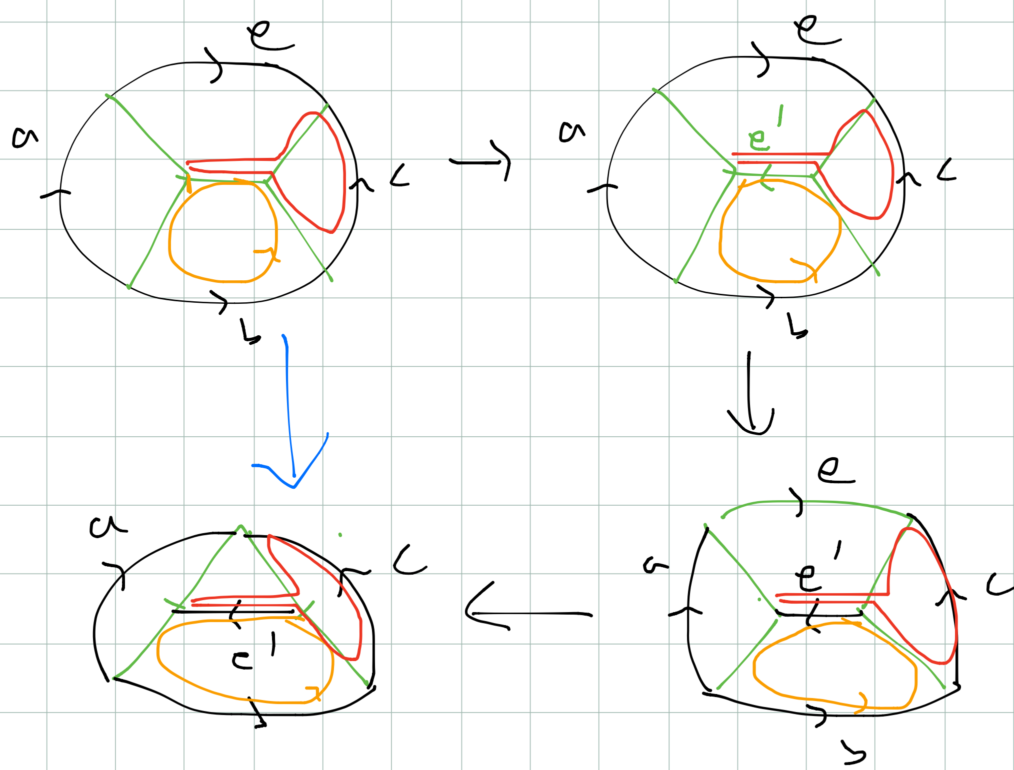
If an edge is in T_i, no problem, get a new tree, nothing has happened to F_n

If you want to collapse $e \notin T_i$:
replace e by an (oriented) edge e' of T_i



Different choices of e' give different isos $\pi_1 X_i \cong F_n$

Effect on F_n:



$a \mapsto a$

$b \mapsto be$

$c \mapsto e^{-1}ce$

$e \mapsto e$

multiplies some generators on right or left by $e^{\pm 1}$

$= \rho_{ij}$ or λ_{ij} or both